This document is published in:

Theory and Decision (2003), 54 (2), pp. 163-184.
Doi: http://www.dx.doi.org/10.1023/A:1026256800461

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GENERALIZED EXTERNALITY GAMES

ABSTRACT. Externality games are studied in Grafe et al. (1998, *Math. Methods Op. Res.* 48, 71). We define a generalization of this class of games and show, using the methodology in Izquierdo and Rafels (1996, 2001, Working paper, Univ Barcelona; *Games Econ. Behav.* 36, 174), some properties of the new class of generalized externality games. They include, among others, the algebraic structure of the game, convexity, and their implications for the study of cooperative solutions. Also the proportional rule is characterized for this class of games.

KEY WORDS: Cooperative games, Externality games, Proportional Rule

1. INTRODUCTION

Cooperative games deal with situations in which the different subsets of a set of players (called coalitions) have a well defined expectation of what to get by working alone. The mathematical model requires defining a real function (called characteristic function) that assigns a number to every coalition of players. Then, the formal objective of the cooperative game theory is to give a number to each player taking into account the information available in the characteristic function. Sometimes the solution is a vector (where each component is a player's payoff) and other times it is a set of vectors (including the empty set). The different solutions are often interpreted as different normative proposals to share the benefits of forming the coalition of all players.

Although the study of the consequences of different normative motivations to share profits is by itself an interesting field of research, many authors have stressed the economic applications of the model. This approach has two consequences. First, cooperative game theory will play a role in Economics as long as some interesting economic situations can be modeled via a characteristic function. Second, the characteristic function of a particular economic situation will typically have more structure than more general ones, and this may imply in turn more results and interpretations.

In the literature of cooperative games, there are many examples of characteristic functions obtained after a more primite economic model. Among them, we can list bankruptcy games (see a survey in Thomson, 2002), airport games (Littlechild and Thomson, 1977), market games (Shubik, 1984 and references therein), and, more relevant to our work, externality games (Grafe et al.,1998) and financial games (Izquierdo and Rafels, 1996).

In the present work we define the class of generalized externality games, GEG, which include externality games as defined by Grafe et al. (1998). The characteristic function of GEG can be separated into two functions, one that depends on the totality of the resources belonging to the coalition, and another that depends on the number of members of the coalition. Then, using the methodology in Izquierdo and Rafels (1996, 2001), we study some properties of this new class of games. In particular, we find that each of the families that form the class of GEG has a vectorial space structure, and furthermore, that minimum participation games form an interesting base. Next we show that GEG are semi-convex, but not convex, and show sufficient conditions for convexity. It is also shown that GEG belong to the family of average monotonic games.

The importance of these properties becomes clear when we study different solution concepts. In the spirit of many other works we define a proportional solution for GEG and present an axiomatic characterization. The vectorial space structure of GEG and the fact that minimum participation games constitute a base are used in showing this result. Interestingly enough, the axiomatization of the proportional solution for GEG is the same as for financial games, but is not a generalization of the axiomatization for externality games.

From the above mentioned properties it follows that the core of GEG is non empty, as the proportional solution is always in it, and that the core and the bargaining set coincide. The conditions for convexity are useful if one is interested in GEG for which the Shapley value is in the core. Finally, the property of semi-convexity allows us to use a simple formula for another solution concept, the τ-value.

Finally, we show some examples to illustrate the applications of the mentioned definitions and results. When studying a class of GEG, the provision of a public good, we find the remarkable result that the Shapley value and the proportional solution coincide.

In Section 2 we define generalized externality games and provide some economic examples. In Section 3 we prove some properties. Section 4 characterizes the proportional solution. Section 5 discusses other solution concepts. Section 6 presents some applications and Section 7 concludes.

2. DEFINITION AND EXAMPLES

Externality games were introduced by Grafe et al. (1998) as a class of cooperative games. In this section we present a generalization of these games and show some interesting economic situations that can be interpreted as generalized externality games.

Using conventional notation, Γ_N will denote the set of characteristic form games played by a given set $N = \{1, 2, ..., n\}$ of players. In these games, each subset $S \subset N$ (called a coalition) is associated with a value $v(S)$. Denote by 2^N the set of subsets of N.

DEFINITION 1. A game $v \in \Gamma_N$ is a generalized externality game, GEG, if there exists a vector $\beta = (\beta_i)_{i \in N}$ in $\Re_+^N$, a parameter $\alpha \geq 1$, and a non-decreasing function $r : 2^N \to \Re_+$, such that $v(S) = (\sum_{i \in S} \beta_i)^\alpha r(s)$, where s denotes the cardinal of coalition S. The set of generalized externality games of N players will be denoted by GEG_N.

When $\alpha = 1$, Definition 1 coincides with the definition of externality games. Generalized externality games can be interpreted as a situation in which players contribute both with their endowments (β_i) and their presence (through the function r) to the coalition where they belong. One can easily check that these games are monotone and superadditive. Monotonicity requires that $v(S) \leq v(T)$ whenever $S \subset T$, whereas superadditivity means that $v(S) + v(T) \leq v(S \cup T)$ for all coalitions such that $S \cap T = \emptyset$.

EXAMPLE 1 (Provision of public goods): Consider the following model in Moulin (1992). Let A be a set of public decisions and denote by $c(a)$ the cost of financing decision a. A set of agents, $N = \{1, 2, ..., n\}$, must share the cost of decision a. A feasible outcome is a vector $(a; y_1, ..., y_n)$ where $a \in A$; $\sum_{i \in N} y_i = c(a)$, and y_i is agent i's cost share. Preferences are represented by $u_i(a, y_i)$. Suppose now that we have a quadratic cost function and linear utilities; i.e., $c(a) = a^2/2$ and $u_i(a, y_i) = \beta_i a - y_i$, where the parameter β_i is agent i's marginal rate of substitution between private and public goods. To compute the surplus $v(S)$ generated by coalition S standing alone solve

$$\begin{aligned} \max\nolimits_{a,y_i} \quad & \sum_{i \in S} (\beta_i a - y_i) \\ s.t. \quad & \sum_{i \in S} y_i = a^2/2 \end{aligned}$$

to get $\sum\limits_{i \in S} \beta_i = a$. Then

$$\begin{aligned} v(S) &= \max_{a,y_i} \left(\sum_{i \in S} (\beta_i a) - a^2/2 \right) \\ &= \sum_{i \in S} \left(\beta_i \sum_{i \in S} \beta_i \right) - \left(\sum_{i \in S} \beta_i \right)^2 /2 \\ &= \left(\sum_{i \in S} \beta_i \right)^2 /2, \end{aligned}$$

and a generalized externality game is defined with $r(s) = \frac{1}{2}$ and $\alpha = 2$.

EXAMPLE 2 (Joint venture): Suppose that a set $N = \{1, 2, ..., n\}$ of firms decides to collaborate in a joint venture, and that each firm participates with two factors. One of them, β_i (e.g., labor) is idiosyncratic to each firm and the other, K (e.g., capital) must be equal for all firms. If the technology of the joint venture can be represented by a Cobb–Douglas function of the form $f\left(K, \sum_{i \in N} \beta_i\right) = \left(\sum_{i \in N} \beta_i\right)^\alpha (nK)^\delta$ we can define a cooperative game where $v(S) = \left(\sum_{i \in S} \beta_i\right)^\alpha (sK)^\delta$ if we consider the possibility of coalitions of firms having their own joint venture. Finally, a generalized externality game is defined if $\alpha \geq 1$. In this case $r(s) = (sK)^{\delta}$. Firms may use a solution of this game to decide upon a division of revenues generated by this activity.

3. PROPERTIES OF GENERALIZED EXTERNALITY GAMES.

It is well known that characteristic form games are a vectorial space of dimension 2^{N-1}, and that unanimity games constitute a base of this space. This algebraic structure has proved very useful in the study of cooperative games. It is therefore important to know, for a given class of characteristic form games, whether it preserves the structure of vectorial space and whether one can find an interesting base. In the next proposition we show that this is indeed the case for each one of the subclasses that constitute the class of generalized externality games. To this end we need the following definition. Coalitions will be denoted by upper case letters and their cardinality by the corresponding lower case letter.

DEFINITION 2. A game of minimum participation associated with a vector β and a coalition T is denoted by $v_{T,\beta}$ and defined as:

$$\begin{aligned}
v_{T,\beta}(S) = 0 \quad & \text{if } s < t \text{ or} \\
& \text{if } s = t \text{ and } \sum_{i \in S} \beta_i < \sum_{i \in T} \beta_i, \\
v_{T,\beta}(S) = \sum_{i \in S} \beta_i \quad & \text{if } s \geq t \text{ or} \\
& \text{if } s = t \text{ and } \sum_{i \in S} \beta_i \geq \sum_{i \in T} \beta_i.
\end{aligned}$$

The set of games of minimum participation associated with a generalized externality game v, denoted by v_{β}, is defined by $v_{\beta} = (v_T, \beta)_{T \in 2^N}$.

Minimum participation games are very simple. If a coalition is not big enough, it can guarantee nothing to its members, otherwise it guarantees only the sum of what the members contribute via the parameters β_i.

For a given game consider a maximal set of coalitions satisfying that, for every two coalitions S and T, either their cardinal is different, $s \neq t$, or $\sum_{i \in S} \beta_i \neq \sum_{i \in T} \beta_i$. Then define the set $\mathfrak{L} = \{S_0, S_1, ..., S_m\}$, where (i) $s_{k-1} < s_k$ or (ii) if $s_{k-1} = s_k$, then $\sum_{i \in S_{k-1}} \beta_i < \sum_{i \in S_k} \beta_i$. Coalitions out of this set will be identified with a coalition in the set with the same cardinal and same amount of resources. Now we can state a proposition about the algebraic structure of the GEG_N that will be useful when studying solutions for these games. Denote by $GEG_N(\alpha, \beta)$ the subset of GEG_N with parameter α and with vector β of endowments of coalitions.

PROPOSITION 1. *Given a game in $GEG_N(\alpha, \beta)$, define a set $\mathfrak{L}$ of coalitions as before. Then, the set of minimum participation games associated with coalitions in $\mathfrak{L}$, $v_\beta = (v_{T,\beta})_{T \in \mathfrak{L}}$, form a base of $GEG_N(\alpha, \beta)$.*

Proof. First show that games in $v_\beta = (v_{T,\beta})_{T \in \mathfrak{L}}$ are linearly independent. This means that

$$\sum_{T \in \mathfrak{L}} \lambda_T v_{T,\beta} = 0_N, \tag{1}$$

where 0_N is the vector in $\Re^N$ with a zero in each component, has $\lambda_T = 0$ for all T. Suppose that this is not the case and that there exists a $\lambda_T \neq 0$. Select a coalition $S \in \mathfrak{L}$ such that $\lambda_S \neq 0$, $s \leq t$ for all $T \in \mathfrak{L}$, and whenever $s = t$, $\sum_{i \in S} \beta_i < \sum_{i \in T} \beta_i$. We can rewrite (1) as

$$\begin{aligned} v_{S,\beta}(S) &= \sum_{T \neq S \in \mathfrak{L}} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) \\ &= \sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) + \sum_{T:t>s} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) \\ &\quad + \sum_{\substack{T:t=s \\ \sum_{i \in T} \beta_i > \sum_{i \in S} \beta_i}} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) + \sum_{\substack{T:t=s \\ \sum_{i \in T} \beta_i < \sum_{i \in S} \beta_i}} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) \\ &= \sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) = \sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} \sum_{i \in S} \beta_i \end{aligned}$$

Notice that in the expression in the middle, all terms are zero except for the first. In the second and third, $v_{T,\beta}(S) = 0$ because of the definition of minimum participation games, and in the fourth $\lambda_T = 0$ for all T because of the way S was chosen. From

$$v_{S,\beta}(S) = \sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} \sum_{i\in S} \beta_i$$

we have

$$\sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} = 1 \text{ as } v_{S,\beta}(S) = \sum_{i\in S} \beta_i .$$

But this means that $\lambda_T \neq 0$ for some T, in contradiction with the way S was chosen.

Now we show that every $v \in GEG_N(\alpha, \beta)$ can be written as a linear combination of games of minimum participation. To this end notice that, given any $v \in GEG_N(\alpha, \beta)$, for any $S \subset N$, there exists a coalition $S_h \in \mathfrak{L}$ such that $\sum_{i\in S_h} \beta_i = \sum_{i\in S} \beta_i$ and $s_h = s$. Now consider the linear combination $\sum_{S_k\in\mathfrak{L}} \lambda_{S_k} v_{S_k,\beta}$, with λ_{S_k} defined as

$$\lambda_{S_k} = \frac{v\left(S_k\right)}{\sum\limits_{i\in S_k} \beta_i} - \frac{v\left(S_{k-1}\right)}{\sum\limits_{i\in S_{k-1}} \beta_i}$$

then we have

$$\sum_{S_k\in\mathfrak{L}} \lambda_{S_k} v_{S_k,\beta}\left(S\right) = \sum_{S_k\in\mathfrak{L}} \left[\frac{v\left(S_k\right)}{\sum\limits_{i\in S_k} \beta_i} - \frac{v\left(S_{k-1}\right)}{\sum\limits_{i\in S_{k-1}} \beta_i}\right] v_{S_k,\beta}\left(S\right)$$

$$= \sum_{S_k\in\mathfrak{L}:s_k<s} \left[\frac{v\left(S_k\right)}{\sum\limits_{i\in S_k} \beta_i} - \frac{v\left(S_{k-1}\right)}{\sum\limits_{i\in S_{k-1}} \beta_i}\right] \sum_{i\in S} \beta_i$$

$$+ \sum_{\substack{S_k\in\mathfrak{L}:s_k=s \\ \sum_{i\in S_k}\beta_i \leq \sum_{i\in S}\beta_i}} \left[\frac{v\left(S_k\right)}{\sum\limits_{i\in S_k} \beta_i} - \frac{v\left(S_{k-1}\right)}{\sum\limits_{i\in S_{k-1}} \beta_i}\right] \sum_{i\in S} \beta_i$$

$$+ \sum_{\substack{S_k \in \mathfrak{L}: s_k = s \\ \sum_{i \in S_k} \beta_i > \sum_{i \in S} \beta_i}} \left[\frac{v\left(S_k\right)}{\sum_{i \in S_k} \beta_i} - \frac{v\left(S_{k-1}\right)}{\sum_{i \in S_{k-1}} \beta_i} \right] v_{S_k, \beta}\left(S\right)$$

$$+ \sum_{S_k \in \mathfrak{L}: s_k > s} \left[\frac{v\left(S_k\right)}{\sum_{i \in S_k} \beta_i} - \frac{v\left(S_{k-1}\right)}{\sum_{i \in S_{k-1}} \beta_i} \right] v_{S_k, \beta}\left(S\right)$$

$$= v\left(S\right)$$

This completes the proof.

This property of generalized externality games will be used in the next section, when the proportional rule is axiomatized for these games.

Another interesting property for cooperative games is convexity as it allows relating different solution concepts. However, convexity is a too strong concept for many purposes. Weaker versions of this concept have been developed, among them, semiconvexity. Next we show that GEG_N are semiconvex, but not convex.

DEFINITION 3. (Driessen and Tijs, 1985) A cooperative game (N, v) is semiconvex if (i) $v\left(N\right) - v\left(N\setminus\{i\}\right) \geq v\left(\{i\}\right)$, and (ii) $v\left(S\right) - \sum_{j \in S\setminus\{i\}}\left(v\left(N\right) - v(N\setminus\{j\})\right) \leq v\left(\{i\}\right)$ for all individuals and coalitions.

PROPOSITION 2. *Generalized externality games are semiconvex.*

Proof. To show that (i) in the definition is satisfied recall that $\beta_i^{\alpha} r\left(1\right) \leq \beta_i^{\alpha} r\left(n\right)$. Then

$$\beta_i^{\alpha} r\left(1\right) \leq \left(\sum_{j \in N\setminus i} \beta_j \right)^{\alpha} \left(r\left(n\right) - r\left(n-1\right)\right)$$
$$+ \binom{\alpha}{1} \left(\sum_{j \in N\setminus i} \beta_j \right)^{\alpha - 1} \beta_i r\left(n\right) + ... + \binom{\alpha}{\alpha} \beta_i^{\alpha} r\left(n\right)$$

$$= \left(\sum_{i\in N}\beta_i\right)^{\alpha} r\,(n) - \left(\sum_{j\in N\setminus i}\beta_j\right)^{\alpha} r\,(n-1)\,.$$

To show (ii):

$$\beta_i^{\alpha} r\,(1) \geq \left(\sum_{j\in S}\beta_j\right)^{\alpha} r\,(s) - (s-1)\left(\sum_{i\in S}\beta_i + \sum_{i\in N\setminus S}\beta_i\right)^{\alpha} r\,(n)$$

$$- \sum_{j\in S\setminus i}\left(\sum_{i\in N\setminus j}\beta_i\right)^{\alpha} r\,(n-1)$$

$$= \left(\sum_{j\in S}\beta_j\right)^{\alpha} r\,(s) - (s-1)\left(\sum_{i\in N}\beta_i\right)^{\alpha} r\,(n)$$

$$- \sum_{j\in S\setminus i}\left(\sum_{i\in N\setminus j}\beta_i\right)^{\alpha} r\,(n-1)$$

$$= \left(\sum_{j\in S}\beta_j\right)^{\alpha} r\,(s) - \sum_{j\in S\setminus i}\left(\sum_{i\in N}\beta_i\right)^{\alpha} r\,(n)$$

$$- \left(\sum_{i\in N\setminus j}\beta_i\right)^{\alpha} r\,(n-1)\,.$$

This property of GEG_N will be helpful in providing a simple formula to compute the τ-value, a cooperative solution.

Generalized externality games are not convex in general, as shown in an example below. Next we show a sufficient condition for a class of generalized externality games to be convex.

DEFINITION 4. A cooperative game (N, v) is convex if $v(S \cup \{i, j\}) - v(S \cup \{j\}) \geq v\,(S \cup \{i\}) - v\,(S)$ for all $S \subset N$, $i, j \notin S$.

The next proposition shows that if the function $r\,(.)$ increases sufficiently fast, then the game is convex.

PROPOSITION 3. *Let (N, v) be a symmetric generalized externality game with $\alpha \in \mathcal{N}$, then, if*

$$\frac{r\,(s+1)}{r\,(s)} > 2,$$

the game v is convex.

Proof. Recall that symmetry means that $v(S \cup \{j\}) = v\,(S \cup \{i\})$ for all $S \subset N$, $i, j \in N$. The condition of convexity for symmetric games can be written as

$$2v\,(S \cup \{i\}) - v\,(S) \leq v(S \cup \{i, j\}) \text{ for all } i, j \notin S. \tag{2}$$

In the case of generalized externality games symmetry implies $\sum_{i \in S} \beta_i = \sum_{i \in T} \beta_i$ whenever $s = t$, and we can write

$$\begin{aligned} &2v\,(S \cup \{i\}) - v\,(S) \\ &= 2\left(\sum_{k \in S} \beta_k + \beta_i\right)^{\alpha} r\,(s+1) - \left(\sum_{k \in S} \beta_k\right)^{\alpha} r\,(s) \end{aligned}$$

and

$$\begin{aligned} v(S \cup \{i, j\}) = \Bigg[&\left(\sum_{k \in S} \beta_k\right)^{\alpha} + \binom{\alpha}{1} \left(\sum_{k \in S} \beta_k\right)^{\alpha - 1} (\beta_i + \beta_j) + \ldots \\ &\ldots + (\beta_i + \beta_j)^{\alpha} \Bigg] r\,(s+2)\,. \end{aligned}$$

To verify (2) write

$$\begin{aligned} &2\left(\sum_{k \in S} \beta_k + \beta_i\right)^{\alpha} \frac{r\,(s+1)}{r\,(s+2)} - \left(\sum_{k \in S} \beta_k\right)^{\alpha} \frac{r\,(s)}{r\,(s+2)} \\ &= \left(\sum_{k \in S} \beta_k\right)^{\alpha} \frac{2r\,(s+1) - r\,(s)}{r\,(s+2)} \\ &\quad + \left[\binom{\alpha}{1} \left(\sum_{k \in S} \beta_k\right)^{\alpha - 1} (\beta_i) + \ldots \right] \frac{2r\,(s+1)}{r\,(s+2)} \\ &\leq \left(\sum_{k \in S} \beta_k\right)^{\alpha} \frac{2r\,(s+1) - r\,(s)}{r\,(s+2)} \end{aligned}$$

$$+2\binom{\alpha}{1}\left(\sum_{k\in S}\beta_k\right)^{\alpha-1}(\beta_i)\,\frac{r\,(s+1)}{2r\,(s+2)}+\ldots$$

$$\leq\left[\left(\sum_{k\in S}\beta_k\right)^{\alpha}+\binom{\alpha}{1}\left(\sum_{k\in S}\beta_k\right)^{\alpha-1}(\beta_i+\beta_j)+\ldots\right.$$

$$\left.\ldots+(\beta_i+\beta_j)^{\alpha}\right]=\frac{v(S\cup\{i,j\})}{r\,(s+2)}$$

as required by convexity.

Counterexample: Consider the generalized externality game defined by $N=\{1,2,3\}$, $\beta=(1,2,20)$, $\alpha=2$, and $r\,(1)=1$, $r\,(2)=3$, and $r\,(3)=4$. This game is not convex as, for example, $v\,(\{3\}\cup\{2,1\})-v\,(\{3\}\cup\{1\})=793$, whereas $v\,(\{3\}\cup\{2\})-v\,(\{3\})=1,052$.

Another property of interest relates GEG with the class of average monotonic games (Izquierdo and Rafels, 2001). This is formalized in the next proposition.

DEFINITION 5. (Izquierdo and Rafels, 2001) A cooperative game (N,v) is average monotonic if
(i) $v\,(S)\geq 0$ for all coalitions $S\subset N$, and
(ii) there exists a vector $\alpha\in\Re^N_+\setminus\{0\}$ such that $(\sum_{i\in T}\alpha_i)v\,(S)\leq(\sum_{i\in S}\alpha_i)v\,(T)$ for $S\subset T\subset N$.

PROPOSITION 4. *GEG are average monotonic.*

Proof. To show (i) in Definition 5 see that $\beta_i\geq 0$ and $r\,(s)\geq 0$ imply $v\,(S)\geq 0$. To show (ii) let $\alpha=\beta$. Then

$$\frac{v\,(S)}{\sum_{i\in S}\alpha_i}=\frac{\left(\sum_{i\in S}\beta_i\right)^{\alpha}r\,(s)}{\sum_{i\in S}\beta_i}=\left(\sum_{i\in S}\beta_i\right)^{\alpha-1}r\,(s)$$

$$\leq\left(\sum_{i\in T}\beta_i\right)^{\alpha-1}r\,(t)=\frac{v\,(T)}{\sum_{i\in T}\alpha_i}.$$

The inequality holds because both r and $x^{\alpha-1}$ with $\alpha \geq 1$ are increasing functions.

4. THE PROPORTIONAL SOLUTION

Proportional solutions have been suggested in many contexts, like bankruptcy problems (see O'Neill, 1982; Chun, 1988; Thomson, 1995). Typically, in cooperative games, the proportional solution is presented as giving each player the part of what the grand coalition can get that is proportional to its value. I.e.,

$$p_i\left(v\right) = \frac{v\left(i\right)}{\sum\limits_{j \in N} v\left(j\right)} v\left(N\right).$$

However, this is not the only possibility. In games with more structure, there may be other elements that are good candidates to define proportionality (claims, initial endowments,...). For GEG we provide the following definition.

DEFINITION 6. (Adapted from Izquierdo and Rafels, 2001, to GEG). Let (N, v) be a generalized externality game with $\beta = (\beta_1, ..., \beta_n)$ as players' endowments, then the proportional solution, $p(v, \beta) \in \Re^n$ is defined as

$$p(v, \beta) = (p_i(v, \beta))_{i \in N} = \left(\frac{\beta_i}{\sum\limits_{i \in N} \beta_i} v\left(N\right) \right)_{i \in N}.$$

Grafe et al. (1998) define the proportional rule for externality games as $\Pi(v, \beta) = (\Pi_i(v, \beta))_{i \in N} = (\beta_i v\left(N\right))_{i \in N}$. For these games, the definition above gives

$$\begin{aligned} p_i(v, \beta) &= \frac{\beta_i}{\sum\limits_{i \in N} \beta_i} v\left(N\right) = \frac{\beta_i}{\sum\limits_{i \in N} \beta_i} \sum_{i \in N} \beta_i v\left(n\right) = \beta_i v\left(n\right) \\ &= \Pi_i(v, \beta). \end{aligned}$$

Thus Definition 6 generalizes the proportional solution for externality games.

Izquierdo and Rafels (1996) and Grafe et al. (1998) present an axiomatic characterization of the proportional solution for financial games (a subset of average monotonic games) and externality games, respectively. We show that, for GEG, the characterization in Izquierdo and Rafels (1996) applies, but that the one in Grafe et al. (1998) does not. It is easy to show that generalized externality games satisfy the following properties (listed in Izquierdo and Rafels (1996)).

- Individual pseudo-rationality (IPR): if $v(N) \geq \sum_{i \in N} \beta_i$, then $p_i(v, \beta) \geq \beta_i$. This means that, if the grand coalition can get more than the total amount of endowments provided by the individuals, each player gets, at least, her endowment.
- Efficiency (EF): $\sum_{i \in N} p_i = v(N)$.
- Restricted linearity (RL): Let v_1 and v_2 be two games in GEG_N with the same vector β, then
  - (i) if v_1+v_2 is a GEG_N with the same vector β, then $p_i(v_1, \beta) + p_i(v_2, \beta) = p_i(v_1 + v_2, \beta)$, and
  - (ii) $p_i(\lambda v_1, \beta) = \lambda p_i(v_1, \beta)$ for all $\lambda \in \Re^+$.

Restricted linearity will be discussed later, when we compare the proportional solution with other solutions.

The next proposition shows the sufficiency of these properties to characterize the proportional solution for GEG_N.

PROPOSITION 5. *The proportional solution is the only solution that satisfies IPR, EF and RL within the set of* GEG_N.

Proof. Let $v \in GEG_N$, with $v(N) \geq \sum_{i \in N} \beta_i$, and consider a solution Φ^n that satisfies IPR, EF and RL, then we show that it coincides with the proportional solution. By Proposition 1 v can be expressed as $v = \sum_{k=1}^{m} \lambda_{S_k} v_{S_k,\beta}$, where

$$\lambda_{S_k} = \frac{v(S_k)}{\sum\limits_{i \in S_k} \beta_i} - \frac{v(S_{k-1})}{\sum\limits_{i \in S_{k-1}} \beta_i},$$

and $v_{S_k,\beta}$ are minimum participation games.

Using properties IPR and RL we can write

$$\begin{aligned}\Phi_i(v, \beta) &= \Phi_i\left(\sum_{k=1}^{m} \lambda_{S_k} v_{S_k,\beta}, \beta\right) = \sum_{k=1}^{m} \lambda_{S_k} \Phi_i(v_{S_k,\beta}, \beta) \\ &\geq \sum_{k=1}^{m} \lambda_{S_k} \beta_i = \frac{v(N)}{\sum_{i \in N} \beta_i} \beta_i = p_i(v, \beta).\end{aligned}$$

By EF of the proportional solution it must be $\Phi_i(v, \beta) = p_i(v, \beta)$ for all $i \in N$.

For games with $v(N) < \sum_{i \in N} \beta_i$ define a new game $v' = \lambda v$ with $\lambda \in \Re^+$ such that $v'(N) \geq \sum_{i \in N} \beta_i$. For this game we have $\Phi_i(v', \beta) = p_i(v', \beta)$. By (ii) in RL it must be that $\Phi_i(v, \beta)$ $p_i(v, \beta)$.

Grafe et al. (1998) provide a characterization of the proportional solution for externality games (GEG with $\alpha = 1$). However, this result cannot be generalized to generalized externality games. The axioms in Grafe et al.(1998) for a solution Φ on EG_N are:

- Individual rationality (IR): for all v in EG_N, $\Phi_i(v) \geq \beta_i r(1)$.
- Monotonicity (M): for all $v(\beta, r)$ and $v(\beta, r')$ in EG_N: if $r(t) \leq r'(t)$ for all $t \in \{1, ..., n\}$ then $\Phi_i(v(\beta, r)) \leq \Phi_i(v(\beta, r'))$.
- Efficiency (EF): As before.

It is straightforward to show that the proportional solution verifies these axioms for the class of generalized externality games. However, it is not characterized by them, as there are other solutions that satisfy the same set of axioms. For instance, take the solution Ψ defined by

$$\Psi_i = \frac{\beta_i^\alpha}{\sum_{j \in N} \beta_j^\alpha} v(N).$$

This solution clearly satisfies EF and M. To show that it also satisfies IR write

$$\Psi_i = \frac{\beta_i^\alpha}{\sum_{j\in N}\beta_j^\alpha} v(N) = \frac{\beta_i^\alpha}{\sum_{j\in N}\beta_j^\alpha}\left(\sum_{j\in N}\beta_j\right)^\alpha r(n)$$

$$\geq \frac{\beta_i^\alpha}{\sum_{j\in N}\beta_j^\alpha}\left(\sum_{j\in N}\beta_j\right)^\alpha r(1) = \frac{\left(\sum_{i\in N}\beta_j\right)^\alpha}{\sum_{j\in N}\beta_j^\alpha} v(i) \geq v(i).$$

The last inequality holds because $(\sum_{i\in N}\beta_j)^\alpha \geq \sum_{j\in N}\beta_j^\alpha$ for $\alpha \geq 1$.

5. OTHER SOLUTIONS OF GEG_N.

Izquierdo and Rafels (2001) show that the core of average monotonic games is non empty and that it contains the proportional solution. They also show that the core coincides with the two most important definitions of bargaining set presented in Aumann and Maschler (1964) and in Mas-Colell (1989). Since generalized externality games are average monotonic, the same properties apply. The reader is referred to the mentioned works for definitions and details on the core and bargaining sets.

Grafe et al. (1998) show an example of an externality game (and, *a fortiori*, a GEG) where the Shapley value is not in the core. When the game is convex, the Shapley value is in the core. Proposition 4 provided a sufficient condition for GEG to be convex. Next we define the Shapley value for the sake of completeness, and because it will be used in the next section, where its relation with the proportional solution will be explored.

DEFINITION 7. The Shapley value is defined as the only solution $Sh(v)$ that satisfies the following four properties:

(i) Symmetry (SYM): if $v(S\cup\{i\}) = v(S\cup\{j\})$ for all $S \subset N - \{i, j\}$, then $\Phi_i(v) = \Phi_j(v)$,

(ii) Additivity (ADD): $\Phi(v + w) = \Phi(v) + \Phi(w)$,
(iii) Eficiency (EF): $\sum_{i \in N} \Phi_i(v) = v(N)$, and
(iv) Dummy player (DP): if $v(S \cup \{i\}) = v(S)$ a for all $S \subset N$ then $\Phi_i(v) = 0$.

It can be shown that

$$Sh_i(v) = \sum_{i \notin T} \frac{t!\,(n-t-1)!}{n!} \left[v\left(T \cup \{i\}\right) - v\left(T\right) \right]. \qquad (3)$$

Tijs (1981) proposes the solution concept called the τ-value. The motivation for this value is that it represents a compromise among players, as it gives every player a payoff between a superior and an inferior bound. The superior bound is defined as $M^v = (M_i^v)_{i \in N}$, with $M_i^v = v\left(N\right) - v\left(N \backslash \{i\}\right)$, while the inferior bound is $m^v = (m_i^v)_{i \in N}$, with $m_i^v = \max_{S \subset N \backslash \{i\}} [v(S \cup \{i\} - \sum_{j \in S} M_j^v)]$. See that the superior bound has the marginal contribution of every player to the grand coalition, and that the inferior bound has the minimum payoff that players have after the other players in the coalition are given their superior bound. The τ-value is defined only for quasi-equilibrated games. These are games that satisfy $\sum_{i \in S} m_i^v \leq v\left(N\right) \leq \sum_{i \in S} M_i^v$ and $m_i^v \leq M_i^v$ for all $i \in N$. Games with a non-empty core are quasi-equilibrated. Thus the τ-value is well defined for GEG.

DEFINITION 8. The τ-value of a quasi-equilibrated game is defined as the only efficient point that lies on the segment joining the superior and inferior bounds.

Driessen and Tijs (1985) show that, for games with a non-empty core, the $\tau-$value can be computed using the formula $\tau(v) = (1 - \delta)\, m^v + \delta M^v$, where

$$\delta = \frac{v\left(N\right) - \sum_{i \in N} m_i^v}{\sum_{i \in S} M_i^v - \sum_{i \in N} m_i^v}.$$

This formula can be used for games in GEG_N.

Driessen and Tijs (1983) also show that, for balanced semiconvex games with at most four players, the τ-value belongs to the core. As generalized externality games are both balanced and semiconvex, the same applies for these games whenever $N \leq 4$. For the general case, Driessen and Tijs (1985) provide a necessary and sufficient condition for the τ-value to belong to the core in semiconvex games, which include GEG_N.

6. EXAMPLES AND APPLICATIONS

Consider, again, the game of provision of public goods (Example 1 in Section 2). In this section we provide some examples of this class of games, and find the proportional solutions and the τ-value. We also show that, for these games, the Shapley value and the proportional solution coincide. Next we show some examples of joint ventures (Example 2 in Section 2).

EXAMPLE 3. Provision of public goods: $v(S) = \left(\sum_{i \in S} \beta_i\right)^2 / 2$

Case 1.1: Two players, $\beta = (1, 2)$. In this case, $v(1) = 0.5$, $v(2) = 2$, $v(1, 2) = 4.5$. The proportional solution is $p_1 = \frac{1}{3}4.5 = 1.5$, and $p_2 = \frac{2}{3}4.5 = 3$. We can compare this solution with the Lindahl solution for public goods, which requires solving the following problems.

$$\begin{array}{ll} \max_{a, y_i} & u_i = \beta_i a^d - y_i \\ \text{s.t.} & \pi_i a^d = \pi_y y_i + \lambda_i \Pi_{\max} \end{array}$$

for $i = 1, 2$, and

$$\begin{array}{ll} \max_{a, y} & \Pi = (\pi_1 + \pi_2)\, a^s - \pi_y y \\ \text{s.t.} & y = \frac{a^2}{2}. \end{array}$$

In these problems, π_i is the personalized price that Player i pays to consume the public good, π_y is the price of the public good, $\lambda_1 + \lambda_2 = 1$, and $\Pi_{\max}$ is Π evaluated at the maximum. Market clearing conditions are $a^s = a^s$, $y_1 + y_2 = y$. The solution for this problem when $\lambda_1 = 0$ and $\lambda_2 = 1$ is $(a, y_1, y_2) = (3, 3, 1.5)$, with $\left(\pi_1, \pi_2, \pi_y\right) = (1, 2, 1)$, and $(u_1, u_2) = (0, 4.5)$. On the other hand, when $\lambda_1 = 1$ and $\lambda_2 = 0$ we get $(a, y_1, y_2) = (3, -1.5, 6)$, with $\left(\pi_1, \pi_2, \pi_y\right) = (1, 2, 1)$, and $(u_1, u_2) = (4.5, 0)$. Any convex combination between vectors $(0, 4.5)$, and $(4.5, 0)$ can be obtained for other values of λ_1 and λ_2. The proportional solution $p(v) = (1.5, 3)$ is obtained if $(\lambda_1, \lambda_2) = \left(\frac{1}{3}, \frac{2}{3}\right)$. In the Lindahl solution, the technology is only available to the owners of the firm, who are the players themselves, but in different proportions. On the other hand, in the proportional solution it is implicitly assumed that the technology is available to all players and coalitions, thus imposing a minimum utility for the players. This is the main reason for the difference in the solutions.

Case 1.2. Three players, $\beta = (1, 2, 3)$. In this case $v(1) = 0.5$, $v(2) = 2$, $v(3) = 4.5$, $v(1, 2) = 4.5$, $v(1, 3) = 8$, $v(2, 3) = 12.5$, and $v(1, 2, 3) = 18$. The proportional solution is $p(v) = (3, 6, 9)$. In this example we calculate the τ-value using the formula in the previous section and compare the two solutions.

First calculate the vector of superior bounds,

$$\begin{aligned} M^v &= (18 - 12.5, 18 - 8, 18 - 4.5) \\ &= (5.5, 10, 13.5). \end{aligned}$$

To calculate the vector of inferior bounds, m^v, we have

$$\begin{aligned} m_1^v &= \max_{S \subset N \setminus \{1\}} \left[v \left(S \cup \{1\} - \sum_{j \in S} M_j^v \right) \right] \\ &= \max\{18 - (10 + 13.5), 4.5 - 10, 8 - 13.5\} \\ &= -5, 5. \end{aligned}$$

Similarly, $m_2^v = -1$, and $m_3^v = 2.5$. Then $\delta = \frac{18-(-4)}{5.5+10+13.5-(-4)} = \frac{2}{3}$, and

$$\begin{aligned} \tau(v) &= \frac{1}{3}(-5.5, -1, 2.5) + \frac{2}{3}(5.5, 10, 13.5) \\ &= (1.833, 6.333, 9.833). \end{aligned}$$

Case 1.3. Three players, $\beta = (2, 2, 3)$. In this case $v(1) = 2$, $v(2) = 2$, $v(3) = 4.5$, $v(1, 2) = 8$, $v(1, 3) = 12.5$, $v(2, 3) = 12.5$, and $v(1, 2, 3) = 24.5$. The proportional solution is $p(v) = (7, 7, 10.5)$, while the τ-value is $\tau(v) = (6.666, 6.666, 11.166)$.

The last two cases suggest that the proportional solution tends to provide a more egalitarian distribution. Although we do not have a general proof of this result, many other examples agree with it.

One may wonder about the performance of the Shapley value in this class of games. The next proposition shows the remarkable result that it coincides with the proportional solution.

PROPOSITION 6. *Let v be a GEG with $\alpha = 2$ and $r(s) = k$. Then $Sh(v) = p(v)$.*

Proof. First see that $Sh(v)$ satisfies EF by definition. Also notice that ADD implies RL Then it only remains to show that the Shapley value satisfies IPR for this class of games. I.e., we have to show that if $v(N) \geq \sum_{i \in N} \beta_i$, then $Sh_i(v, \beta) \geq \beta_i$.

We first prove that IPR is indeed satisfied when $k = 1$. In this case $v(N) \geq \sum_{i \in N} \beta_i$ means $\left(\sum_{i \in N} \beta_i\right)^2 \geq \sum_{i \in N} \beta_i$, which is equivalent to $\sum_{i \in N} \beta_i \geq 1$.

The formula for the Shapley value in (3) can be written as

$$Sh_i(v) = \sum_{i \notin T} \frac{t!\,(n-t-1)!}{n!} \frac{\left[(v(T \cup \{i\}) - v(T)) + \left(v\left(T^C \cup \{i\}\right) - v\left(T^C\right)\right)\right]}{2},$$

where T^C is $N \backslash (T \cup \{i\})$. This is true because $t! = \left(n - t^C - 1\right)!$, and $(n-t-1)! = t^C!$, where t^C is the cardinal of coalition T^C.

Now we have

$$\frac{1}{2}(v(T \cup \{i\}) - v(T)) + \left(v\left(T^C \cup \{i\}\right) - v\left(T^C\right)\right)$$

$$= \frac{1}{2}\left[\left(\sum_{k \in T \cup \{i\}} \beta_k\right)^2 - \left(\sum_{k \in T} \beta_k\right)^2 + \left(\sum_{k \in T^C \cup \{i\}} \beta_k\right)^2 - \left(\sum_{k \in T^C} \beta_k\right)^2\right]$$

$$= \beta_i^2 + \left(\sum_{k \in T} \beta_k\right)\beta_i + \left(\sum_{k \in T^C} \beta_k\right)\beta_i$$

$$= \beta_i^2 + \left(\sum_{k \in N \backslash \{i\}} \beta_k\right)\beta_i$$

$$= \beta_i \left(\beta_i + \sum_{k \in N \backslash \{i\}} \beta_k\right) = \beta_i \sum_{k \in N} \beta_k.$$

The last term is greater or equal than β_i as long as $\sum_{k\in N}\beta_k \geq 1$. In this case, the Shapley value of Player i is a weighted average of terms greater than or equal to β_i, thus $Sh_i(v) \geq \beta_i$ whenever $v(N) \geq \sum_{i\in N}\beta_i$, as required by IPR.

The fact that the Shapley value and the proportional solution coincide when $k = 1$ along with RL imply that the two concepts also coincide for all $k \geq 0$. This completes the proof.

Another way to read the last proposition is that, for this class of games, the proportional solution provides a very simple formula to compute the Shapley value. Both the proportional solution and the Shapley value are computed using a linear formula, and both are characterized by a linear property (ADD in case of the Shapley value and RL in the case of the proportional solution). However there is no way to identify *a priori* the appropriate version of linearity that characterizes a given solution.

The next example shows that the coincidence between the Shapley value and the proportional solution cannot be generalized for the whole class of GEG.

EXAMPLE 4. *Joint venture:* $v(S) = \left(\sum_{i\in S} L_i\right)^{\alpha}(sK)^{\delta}$.

Case 2.1. Three players, $\beta = L = (1, 1, 2)$, $\alpha = 2$, $K = 1$, $\delta = 1$. Then $v(1) = v(2) = 1$, $v(3) = 2$, $v(1, 2) = 4$, $v(1, 3) = v(2, 3) = 6$, and $v(1, 2, 3) = 12$. The proportional solution is $p(v) = (3, 3, 6)$, while the Shapley value is $Sh(v) = \left(\frac{21}{6}, \frac{21}{6}, \frac{30}{6}\right)$.

7. CONCLUSION

We have defined GEG as a generalization of externality games. The different families of GEG have the structure of a vectorial space with minimum participation games as a base. This property makes an interesting characterization of the proportional solution possible. The relations between other properties of GEG and solutions are also explored. One interesting feature that may deserve more attention is the relation between GEG and financial games, as they share many properties, although the two classes of games are not related by inclusion.

ACKNOWLEDGEMENTS

The author gratefully acknowledges financial support from DGI grant BEC2002-03715 (Ministerio de Ciencia y Tecnología).

REFERENCES

Aumann, R. and Maschler, M. (1964), *The Bargaining Sets for Cooperative Games*. Princeton University Press, Princeton, NJ 443–476.

Chun, Y. (1988), The proportional solution for right problems, *Mathematical Social Sciences* 15, 231–246.

Driessen, T.S.H. and Tijs, S.H. (1985), The τ-value, the core and semiconvex games, *International Journal of Game Theory* 15, 229–248.

Grafe, F., Iñarra, E. and Zarzuelo, J.M. (1998), Population monotonic allocation schemes on externality games, *Mathematical Methods of Operation Research* 48(1), 71–80.

Izquierdo, J.M. and Rafels C. (1996), A generalization of the bankruptcy game: Financial cooperative games. Working Paper. Universidad de Barcelona.

Izquierdo, J.M. and Rafels, C. (2001), Average monotonic cooperative games, *Games and Economic Behavior* 36, 174–192.

Mas-Colell, A. (1989), An equivalence theorem for bargaining sets, *Journal of Mathematical Economics* 18, 129–139.

Littlechild, S.C. and Thomson, G.F. (1977), Aircraft landing fees: a game theory approach, *The Bell Journal of Economics* 8, 186–204.

Moulin, H. (1992), All sorry to disagree: a general principle for the provision of nonrival goods, *Scandinavian Journal of Economics* 94(1), 37–51.

O'Neill, B. (1982), A problem of rights arbitration from the Talmud, *Mathematical Social Sciences* 2, 345–371.

Shubik, M. (1984), *A Game-Theoretic Approach to Political Economy.* The MIT Press, Cambridge, MA.

Thomson, W. (1995), Bankruptcy and taxation problems: an axiomatic analysis. University of Rochester *Working Paper*. Forthcoming in *Mathematical Social Sciences*.

Thomson, W. (2002), Axiomatic and Game-Theoretic Analysis of Bankruptcy and Taxation Problems: A Survey. *mimeo,* University of Rochester.

Tijs, S.H. (1981), Bounds for the core and the τ-value, in: Moeschin, O. and Pallasche, D. (eds.), *Game Theory and Mathematical Economics*, North Holland, pp. 123–132.